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ELLIPTIC BOUNDARY VALUE PROBLEMS IN THE SPACE OF DISTRIBUTIONS

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Introduction

Elliptic boundary value problems have their own long history. For the general system they were, however, first clearly formulated microlocally by M. Kashiwara and T. Kawai [K-K]. Their theorem has enjoyed many applications, for example, to solvability of operators of simple characteristics, hypoelliptic operators, and tangential Cauchy-Riemann systems. The theorem does not give, however, much information if we restrict ourselves in the space of distributions. This note aims at giving an analogous theorem of Kashiwara-Kawai type in case function spaces are tempered. See Theorem 3 in Section 1 for the main theorem. By this theorem, we can obtain many application to distribution boundary values of holomorphic functions (e.g. M. Uchida[U]). The result of this note was obtained while the second author was staying in Univ. de Paris VI and Univ. Paris XIII.

1. Main theorem

Let M be a real analytic manifold of dimension n with a complex neighborhood X . Let \mathcal{M} be a coherent \mathcal{D}_X module on X and assume that \mathcal{M} is elliptic on M , *i.e.*

$$(1) \quad \text{char}(\mathcal{M}) \cap T_M^*X \subset T_X^*X.$$

Let N be a real analytic submanifold of M of codimension $d \geq 1$ in M , and Y be a complexification of N in X . We assume that Y is non-characteristic for \mathcal{M} , *i.e.*

$$(2) \quad \text{char}(\mathcal{M}) \cap T_Y^*X \subset T_X^*X.$$

In this situation, we have the canonical morphisms

$$T_N^*M \xleftarrow[\rho]{} T_N^*X \xrightarrow[\varpi]{} T_X^*X.$$

Under the above notation we have

THEOREM 1. *The natural morphism*

$$\mathbf{R}\rho_* \mathbf{R}\underline{\text{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|X}^f) \leftarrow \mathbf{R}\underline{\text{Hom}}_{\mathcal{D}_X}(\mathcal{M}, T^{-\mu_N}(\mathcal{D}b_M)) \otimes or_{N/M}$$

is an isomorphism.

In the above theorem $or_{N/M}$ denotes the relative orientation sheaf of N in M . The sheaf $\mathcal{C}_{N|X}^f$ on T_N^*X is the tempered version of $\mathcal{C}_{N|X}$ and is given, with the tempered microlocalization due to E. Andronikof[An], by

$$\mathcal{C}_{N|X}^f := T^{-\mu_N}(\mathcal{O}_X) \otimes or_M[n].$$

We remark that the above object in the derived category is concentrated in degree 0. For a point $\overset{\circ}{x} \in T_N^*X$, the stalk of $\mathcal{C}_{N|X}^f$ at $\overset{\circ}{x}$ is given, with the aid of local cohomology with bounds, by

$$\mathcal{C}_{N|X, \overset{\circ}{x}}^f \simeq \varinjlim \mathbb{H}_{[Z]}^n(\mathcal{O}_X)_{\pi_X(\overset{\circ}{x})}.$$

Here π_X denotes the projection $\pi_X : T^*X \rightarrow X$ and the inductive limit is taken for all closed subanalytic sets Z in X satisfying the property

$$C_N(Z)_{\pi_X(\overset{\circ}{x})} \subset \{v \in T_N X; \langle \overset{\circ}{x}, v \rangle < 0\} \cup \{0\}.$$

Refer here to Kashiwara-Schapira[K-S2] for the notion of normal cones $C_N(\cdot)$. The sheaf $T-\mu_N(Db_M)$ on T_N^*M is also constructed by E. Andronikof[An]. We just explain that its stalk at $\overset{\circ}{x} \in T_N^*M$ is given by the isomorphism

$$T-\mu_N(Db_M)_{\overset{\circ}{x}} \simeq \varinjlim_Z \Gamma_Z(Db_M)_{\pi_M(\overset{\circ}{x})}.$$

Here the inductive limit is taken for any closed subanalytic set Z in M with the property

$$C_N(Z)_{\pi_M(\overset{\circ}{x})} \subset \{v \in T_N M; \langle \overset{\circ}{x}, v \rangle < 0\} \cup \{0\}$$

($\pi_M: T^*M \rightarrow M$).

Next we give another theorem, which is analogous to Theorem 6.3.1 of Kashiwara-Shapira [K-S1] (refer also to Kashiwara-Kawai[K-K] where we find the theorem of [K-S1] in its original form).

THEOREM 2. *Let $\tilde{\mathcal{M}} = \mathcal{E}_X \otimes_{\pi_X^{-1}\mathcal{D}_X} \pi_X^{-1}\mathcal{M}$. Then the natural morphism*

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\tilde{\mathcal{M}}, \mathcal{C}_{N|X}^f) \leftarrow \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\tilde{\mathcal{M}}, \mathcal{E}_{X \leftarrow Y}) \bigotimes_{\mathrm{End}(\mathcal{E}_{X \leftarrow Y})}^{\mathbf{L}} \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathcal{E}_{X \leftarrow Y}, \mathcal{C}_{N|X}^f)$$

*is an isomorphism outside of $T_N^*X \cap T_Y^*X$. This entails an isomorphism*

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{N|X}^f) \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_{X \leftarrow Y}) \otimes_{p^{-1}\mathcal{E}_Y}^{\mathbf{L}} p^{-1}\mathcal{C}_N^f$$

*on $T_N^*X \setminus T_Y^*X$ where p is the canonical morphism*

$$p: T_N^*X \setminus T_Y^*X \rightarrow T_N^*Y.$$

In the above theorem, the object \mathcal{C}_N^f on T_N^*Y is the sheaf of temperate microfunctions. This is a subsheaf of \mathcal{C}_N and describes microlocal analytic singularities of distributions on N . By the notation of E. Andronikof[An], this sheaf is defined as

$$\mathcal{C}_N^f := T-\mu_N(\mathcal{O}_Y)[n-d] \otimes \mathrm{or}_{N/Y}.$$

The proof of this theorem is essentially the same as in Theorem 6.3.1 of [K-S1] and relies on the division theorem of temperate microfunctions with holomorphic parameters with respect to microdifferential operators. We also remark that only the non-charactericity of Y is utilized in its proof.

By combining the above theorems into one, we get the main theorem of this note. Let q denote the restriction of ρ to $\overset{\circ}{T}_N^*X \setminus T_M^*X$; $q: \overset{\circ}{T}_N^*X \setminus T_M^*X \rightarrow T_N^*M$ and p the projection $\overset{\circ}{T}_N^*X \setminus T_Y^*X \rightarrow \overset{\circ}{T}_N^*Y$. Then we have

THEOREM 3. *We have a canonical isomorphism on $\overset{\circ}{T}_N^* Y$*

$$\mathbf{R}q_* \left(\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\tilde{\mathcal{M}}, \mathcal{E}_{X \leftarrow Y} |_{\overset{\circ}{T}_N^* X}) \overset{\mathbf{L}}{\otimes}_{p^{-1}\mathcal{E}_Y} p^{-1}\mathcal{C}_N^f \right) \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, T-\mu_N(\mathcal{D}b_M)) \otimes or_{N/M}.$$

2. Idea of Proof

What is left to us is now to construct the morphism in Theorem 1 and to show it an isomorphism.

First we construct a commutative diagram

$$(A) \quad \begin{array}{ccc} \mathbf{R}\rho_! \mathcal{C}_{N|X}^f \otimes or_{N/X} & \longrightarrow & T-\mu_N(\mathcal{A}_M) \\ \downarrow & & \downarrow \\ \mathbf{R}\rho_* \mathcal{C}_{N|X}^f \otimes or_{N/X} & \longleftarrow & T-\mu_N(\mathcal{D}b_M) \end{array}$$

where $T-\mu_N(\mathcal{A}_M)$ is the tempered microlocalization of the sheaf \mathcal{A}_M along N and is constructed by E. Andronikof[A]. This object is the Fourier transform of the tempered specialization $T-\nu_N(\mathcal{A}_M)$ whose stalk at $\overset{\circ}{v} \in T_N M$ is given by

$$T-\nu_N(\mathcal{A}_M)_{\overset{\circ}{v}} \simeq \varinjlim_U \{u \in \mathcal{A}(U); u \text{ is tempered on } M \text{ as a distribution}\}.$$

Here U in the inductive limit ranges through any open subanalytic set in M with the property

$$\overset{\circ}{v} \notin C_N(M \setminus U).$$

To construct (A), it is sufficient to construct its image by the inverse Fourier transformation

$$(A') \quad \begin{array}{ccc} \iota^{-1} T-\nu_N(\mathcal{O}_X) \otimes or_{N/X} & \longrightarrow & T-\nu_N(\mathcal{A}_M) \\ \downarrow & & \downarrow \\ \iota^! T-\nu_N(\mathcal{O}_X) \otimes or_{N/X} & \longleftarrow & T-\nu_N(\mathcal{D}b_M). \end{array}$$

Here ι is the canonical embedding

$$\iota : T_N M \longrightarrow T_N X,$$

and $T-\nu_N(\mathcal{O}_X)$ is the tempered specialization of the sheaf \mathcal{O}_X along N , which is concentrated in degree 0. The stalk of $T-\nu_N(\mathcal{O}_X)$ at $\overset{\circ}{v} \in T_N X$ is given by

$$T-\nu_N(\mathcal{O}_X)_{\overset{\circ}{v}} \simeq \varinjlim_U \{u \in \mathcal{O}(U); u \text{ can be extended to } X \text{ as a distribution}\}$$

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where U runs through all open subanalytic sets in X with $\overset{\circ}{v} \notin C_N(M \setminus U)$. The diagram (A') can be constructed easily if we scrutinize the construction by E. Andronikof[An].

Next we apply $\mathbf{R}\underline{\mathbf{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \cdot)$ to the diagram (A') and obtain the commutative diagram

$$\begin{array}{ccc}
 \mathbf{R}\underline{\mathbf{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \iota^{-1}T^{-\nu_N}(\mathcal{O}_X)) \otimes or_{N/X} & \xrightarrow{\Phi_1} & \mathbf{R}\underline{\mathbf{Hom}}_{\mathcal{D}_X}(\mathcal{M}, T^{-\nu_N}(\mathcal{A}_M)) \\
 \Phi_4 \downarrow & & \downarrow \Phi_2 \\
 \mathbf{R}\underline{\mathbf{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \iota^!T^{-\nu_N}(\mathcal{O}_X)) \otimes or_{N/X} & \xleftarrow{\Phi_3} & \mathbf{R}\underline{\mathbf{Hom}}_{\mathcal{D}_X}(\mathcal{M}, T^{-\nu_N}(\mathcal{D}b_M)).
 \end{array}$$

It is easy to see from the ellipticity of \mathcal{M} that Φ_4 and Φ_2 are isomorphisms. (To show Φ_4 is an isomorphism, it is easier to consider its image by Fourier transformation). Thus to prove that Φ_3 and thus its image by Fourier transformation are isomorphisms, it suffices to show that Φ_1 is an isomorphism. The problem for Φ_1 can be reduced to the case where \mathcal{M} is a single equation; i.e. $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$. Moreover it is sufficient to show that

$$\underline{\mathbf{Hom}}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X P, \iota^{-1}T^{-\nu_M}(\mathcal{O}_X)) \otimes or_{N/X} \longrightarrow \underline{\mathbf{Hom}}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X P, T^{-\nu_N}(\mathcal{A}_M))$$

is surjective. This problem can be solved by using the construction of the elementary solution of P by means of Radon transformation and microdifferential operators.

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